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Common Fixed Point Theorem in Partially OrderedComplete G-Metric space for Six Mappings

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Abstract: In this paper a common fixed point theorem for dominating and weak annihilator mappings in a partially ordered complete G-metric space is proved using continuity, weak compatibilityand compatibility.

Keywords: Partial order, Weakly Compatible maps, Partially ordered G-Metric space.

1.Introduction and preliminaries

The weakly contractive mappings on Hilbert spaces was defined byAlber and Guerre-Delabriere as follows:

Definition 1.1 [2] "A mapping $f : X \to X$ is said to be a weakly contractive mapping if $d(fx, fy) \leq d(x, y) - \varphi(d(x, y))$ for each $x, y \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$."

Theorem 1.2 [9] "Let (X,d)be a complete metric space and $f : X \to X$ be a weakly contractive mapping. Then f has a unique fixed point."

Mustafa and Sims defined G-metric spaces as a generalization of metric space.

Definition 1.3 [8] "Let $G: X \times X \times X \to R^+$ be a function on a non-empty X satisfying

(G-1) $G(x, y, z) = 0$ if $x = y = z$,

(G-2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G-3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),

(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called a generalized metric or more specifically, a G-metric on Xand the pair (X, G) is called a G-metric space."

Zhang and Song defined generalized φ – weak contractive condition as:

Definition 1.4 [10] "Two mappings $T, S : X \to X$ are called generalized φ-weak contractive if there exists a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) = 0$ for $t = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y)) \text{ for each } x, y \in X,$$

where $N(x, y) = \max\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(y, Tx))\}$."

Theorem 1.5 [10] "Let (X,d)be a complete metric space and $T, S : X \to X$ be generalized φ-weak contractive mappings, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ for $t = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then thereexists a unique fixed point $u \in X$ such that $u = Tu = Su$."

The concept of altering distance function was introduced by Khan *et. al* as follows:

Definition 1.6[6] "The function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following conditions hold:

(i) ψ is continuous and non-decreasing;

(ii) $\psi(t) = 0$ if and only if $t = 0$".

Definition 1.7A partial order is a binary relation $\preccurlyeq$over a set X which is reflexive, anti-symmetric and transitive, i.e. which satisfies, for all $p, q, r \in X$;

(i) $p \preccurlyeq p$, (reflexivity)

(ii) If $p \preccurlyeq q$ and $q \preccurlyeq p$ then $p = q$, (anti-symmetry)

(iii) If $p \preccurlyeq q$ and $q \preccurlyeq r$ then $p \preccurlyeq r$. (transitivity)

A set with a partial order$(X,\preccurlyeq)$ is called a partially ordered set.

Definition 1.8 A triplet $(X, G, \preccurlyeq)$ is called a partially ordered G-metric space if $(X,\preccurlyeq)$ is a partially ordered set and (X, G) is a G-metric space.

Definition 1.9[1] "Let $(X, \preccurlyeq)$ be a partially ordered set. A mapping f is called a dominating map on X, if $x \preccurlyeq fx$ for all $x \in X$."

Definition 1.10[1] "Let $(X, \preccurlyeq)$ be a partially ordered set. A mapping f is called a weak annihilator of g, if $fgx \preccurlyeq x$ for all $x \in X$."

Definition 1.11[1] "A subset W of a partially ordered set X is said to be well ordered if every two elements of W be comparable."

Definition 1.12[4]"Let (X,d) be a metric space and f, g: $X \to X$ be two mappings. The pair (f, g) is said to be compatible if and only if

$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$,whenever$\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$for some $t \in X$."

Definition 1.13 [7]"Let(X, G)be a G-metric space and f, g : $X \to X$ be two mappings. The pair (f, g) is said to be compatible if and only if $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \text{ for some } t \in X."$$

Definition 1.14[5]"Let f and g be two self-mappings of a metric space(X, d). Then f and g are said to be weakly compatible if for all $x \in X$, the equality $fx=gx$ implies $fgx = gfx$."

Definition 1.15[3] “Let(X, ≼) be a partially ordered set and f, g, h: X→X be mappings such that f(X)⊆ h(X) and g(X)⊆h(X). The ordered pair (f, g)is said to be partially weakly increasing with respect to h if for all x ∈X,fx≼gy, where
y ∈h^{-1}(fx).”

2. Main Result

Theorem 2.1 Let (X, ≼, G) be a partially ordered complete G-metric space. Let f, g, h, R, S, T : X →X be the six mappings such that f(X) is contained in R(X), g(X) is contained in S(X), h(X) is contained in T(X) and dominating maps f, g and h are weak annihilators of R, S and T respectively. Suppose that for every x, y, z ∈ X,

$$\psi(G(fx, gy, hz)) \leq \psi(M(x, y, z)) - \phi(M(x, y, z)), \quad (2.1)$$

$$\text{where } M(x, y, z)= \left\{ \begin{array}{c} G(Tx,Ry,Sz), G(Ry,Sz,hz), G(Tx,gy,gy), \\ G(Ry,hz,hz), G(Sz,fx,fx), G(Tx,fx,fx), \\ G(Ry,gy,gy), G(Sz,Sz,hz) \end{array} \right\}$$

and ψ, ϕ : [0, ∞) →[0, ∞) are altering distance functions. Then, f, g, h, R, S and T have a unique common fixed point in X provided G-metric space is symmetric and for a non-decreasing sequence $\{x_n\}$ with $x_n \preccurlyeq y_n$ for all n,y_n → u implies that
$x_n \preccurlyeq$ u and one of the following:

(i) g or R and f or T are continuous, (f, T) and (g, R) are compatible and (h, S) is weakly compatible

or

(ii) h or S and f or T are continuous, (f, T) and (h, S) are compatible and(g, R) is weakly compatible

or

(iii) g or R and h or S are continuous, (g, R) and (h, S) are compatible and(f, T) is weakly compatible.

Proof. Let x_0 ∈ X be an arbitrary point. Since f(X) is contained in R(X), we can have x_1∈ X such that fx_0 =Rx_1. Since g(X) is contained in S(X), we can choose x_2∈ X such that gx_1 =Sx_2. Also, as h(X) is contained in T(X), we can choose x_3∈X such that hx_2=Tx_3. Repeating the same argument, we can construct a sequence $\{t_n\}$ defined by

t_{3n+1} = Rx_{3n+1}= fx_{3n},t_{3n+2}= Sx_{3n+2}= gx_{3n+1}andt_{3n+3} = Tx_{3n+3} = hx_{3n+2},
for all n ≥ 0.

Since f, g and h are dominating and f, g and h are weak annihilators of R, S and T, we obtain

$x_0 \preccurlyeq fx_0 = Rx_1 \preccurlyeq fRx_1 \preccurlyeq x_1 \preccurlyeq gx_1$
$= Sx_2 \preccurlyeq gSx_2 \preccurlyeq x_2 \preccurlyeq hx_2$
$= Tx_3 \preccurlyeq hTx_3 \preccurlyeq x_3$.

By continuing this process, we get

$x_1 \preccurlyeq x_2 \preccurlyeq x_3 \preccurlyeq \cdots \preccurlyeq x_k \preccurlyeq x_{k+1} \preccurlyeq \cdots$.

We will complete the proof in three steps.

Step I. We will prove that $\lim_{n\to\infty} G(t_k, t_{k+1}, t_{k+2}) = 0$.

Define $G_k = G(t_k, t_{k+1}, t_{k+2})$. Suppose $G_{k_0} = 0$ for some k_0. Then, $t_{k_0} = t_{k_0+1} = t_{k_0+2}$. Consequently, the sequence $\{t_k\}$ is constant, for $k \geq k_0$. Indeed, let $k_0 = 3n$.

Then $t_{3n} = t_{3n+1} = t_{3n+2}$ and we obtain from (2.1),

$$\psi(G(t_{3n+1}, t_{3n+2}, t_{3n+3})) = \psi(G(fx_{3n}, gx_{3n+1}, hx_{3n+2}))$$
$$\leq \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) - \phi(M(x_{3n}, x_{3n+1}, x_{3n+2})), \quad (2.2)$$

where

$$M(x_{3n}, x_{3n+1}, x_{3n+2})$$
$$= \max\left\{\begin{array}{c} G(Tx_{3n}, Rx_{3n+1}, Sx_{3n+2}), G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\ G(Tx_{3n}, gx_{3n+1}, gx_{3n+1}), \\ G(Rx_{3n+1}, hx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, fx_{3n}, fx_{3n}), \\ G(Tx_{3n}, fx_{3n}, fx_{3n}), \\ G(Rx_{3n+1}, gx_{3n+1}, gx_{3n+1}), G(Sx_{3n+2}, Sx_{3n+2}, hx_{3n+2}) \end{array}\right\} =$$
$$\max\left\{\begin{array}{c} G(t_{3n}, t_{3n+1}, t_{3n+2}), G(t_{3n+1}, t_{3n+2}, t_{3n+3}), \\ G(t_{3n}, t_{3n+2}, t_{3n+2}), \\ G(t_{3n+1}, t_{3n+3}, t_{3n+3}), G(t_{3n+2}, t_{3n+1}, t_{3n+1}), \\ G(t_{3n}, t_{3n+1}, t_{3n+1}), \\ G(t_{3n+1}, t_{3n+2}, t_{3n+2}), G(t_{3n+2}, t_{3n+2}, t_{3n+3}) \end{array}\right\}$$
$$\leq \max\left\{\begin{array}{c} G(t_{3n}, t_{3n+1}, t_{3n+2}), G(t_{3n+1}, t_{3n+2}, t_{3n+3}), \\ G(t_{3n}, t_{3n+1}, t_{3n+2}), \\ G(t_{3n+1}, t_{3n+2}, t_{3n+3}), G(t_{3n}, t_{3n+1}, t_{3n+2}), \\ G(t_{3n}, t_{3n+1}, t_{3n+2}), \\ G(t_{3n+1}, t_{3n+2}, t_{3n+3}), G(t_{3n+1}, t_{3n+2}, t_{3n+3}) \end{array}\right\}$$
$$= G(t_{3n+1}, t_{3n+2}, t_{3n+3}).$$

Now from (2.2),

$\psi(G(t_{3n+1}, t_{3n+2}, t_{3n+3})) \leq \psi(G(t_{3n+1}, t_{3n+2}, t_{3n+3})) - \phi(G(t_{3n+1}, t_{3n+2}, t_{3n+3}))$,

and so, $\phi(G(t_{3n+1}, t_{3n+2}, t_{3n+3})) = 0$, that is, $t_{3n+1} = t_{3n+2} = t_{3n+3}$.

Similarly, if $k_0 = 3n+1$ or $k_0 = 3n+2$, one can easily obtain that

$$t_{3n+2} = t_{3n+3} = t_{3n+4} \text{ and } t_{3n+3} = t_{3n+4} = t_{3n+5}.$$

So the sequence $\{t_k\}$ is constant (for $k \geq k_0$), and t_{k_0} is a common fixed point of R,S, T, f, g and h.

Let for all k, $G_k = G(t_k, t_{k+1}, t_{k+2}) > 0$ (2.3)

We prove that for each k =1, 2, 3,⋯

$G(t_{k+1}, t_{k+2}, t_{k+3}) \leq M(x_k, x_{k+1}, x_{k+2})$

$= G(t_k, t_{k+1}, t_{k+2})$. (2.4)

Let k = 3n. Since $x_{k-1} \preccurlyeq x_k$, using (2.1) we obtain that

$$\psi(G(t_{3n+1}, t_{3n+2}, t_{3n+3})) = \psi(G(fx_{3n}, gx_{3n+1}, hx_{3n+2}))$$
$$\leq \psi(M(x_{3n}, x_{3n+1}, x_{3n+2})) - \phi(M(x_{3n}, x_{3n+1}, x_{3n+2})) \quad (2.5)$$

where

$M(x_{3n}, x_{3n+1}, x_{3n+2})$

$$= \max\left\{\begin{array}{c} G(Tx_{3n}, Rx_{3n+1}, Sx_{3n+2}), G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\ G(Tx_{3n}, gx_{3n+1}, gx_{3n+1}), \\ G(Rx_{3n+1}, hx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, fx_{3n}, fx_{3n}), \\ G(Tx_{3n}, fx_{3n}, fx_{3n}), \\ G(Rx_{3n+1}, gx_{3n+1}, gx_{3n+1}), G(Sx_{3n+2}, Sx_{3n+2}, hx_{3n+2}) \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} G(t_{3n}, t_{3n+1}, t_{3n+2}), G(t_{3n+1}, t_{3n+2}, t_{3n+3}), \\ G(t_{3n}, t_{3n+2}, t_{3n+2}), \\ G(t_{3n+1}, t_{3n+3}, t_{3n+3}), G(t_{3n+2}, t_{3n+1}, t_{3n+1}), \\ G(t_{3n}, t_{3n+1}, t_{3n+1}), \\ G(t_{3n+1}, t_{3n+2}, t_{3n+2}), G(t_{3n+2}, t_{3n+2}, t_{3n+3}) \end{array}\right\}$$

$$\leq \max\left\{\begin{array}{c} G(t_{3n}, t_{3n+1}, t_{3n+2}), G(t_{3n+1}, t_{3n+2}, t_{3n+3}), \\ G(t_{3n}, t_{3n+1}, t_{3n+2}), \\ G(t_{3n+1}, t_{3n+2}, t_{3n+3}), G(t_{3n}, t_{3n+1}, t_{3n+2}), \\ G(t_{3n}, t_{3n+1}, t_{3n+2}), \\ G(t_{3n+1}, t_{3n+2}, t_{3n+3}), G(t_{3n+1}, t_{3n+2}, t_{3n+3}) \end{array}\right\}$$

$= \max\ [G(t_{3n}, t_{3n+1}, t_{3n+2}), G(t_{3n+1}, t_{3n+2}, t_{3n+3})$.

Since ψ is a non-decreasing function, we get

$G(t_{3n+1}, t_{3n+2}, t_{3n+3}) \leq M(x_{3n}, x_{3n+1}, x_{3n+2})$.

If for $n \geq 0$, $G(t_{3n+1}, t_{3n+2}, t_{3n+3}) > G(t_{3n}, t_{3n+1}, t_{3n+2}) > 0$, then

$M(x_{3n}, x_{3n+1}, x_{3n+2}) = G(t_{3n+1}, t_{3n+2}, t_{3n+3})$.

Therefore, (2.5) implies that

$$\psi(G(t_{3n+1}, t_{3n+2}, t_{3n+3})) \leq \psi(G(t_{3n+1}, t_{3n+2}, t_{3n+3})) - \phi(G(t_{3n+1}, t_{3n+2}, t_{3n+3})),$$

which is only possible when $G(t_{3n+1}, t_{3n+2}, t_{3n+3}) = 0$.This is a contradiction to (2.3).

Hence, $G(t_{3n+1}, t_{3n+2}, t_{3n+3}) \leq G(t_{3n}, t_{3n+1}, t_{3n+2})$ and

$M(x_{3n}, x_{3n+1}, x_{3n+2}) = G(t_{3n}, t_{3n+1}, t_{3n+2})$.

Therefore, (2.4) is proved for k = 3n. Similarly, it can be shown that

$G(t_{3n+2}, t_{3n+3}, t_{3n+4}) \leq M(x_{3n+1}, x_{3n+2}, x_{3n+3})$

$= G(t_{3n+1}, t_{3n+2}, t_{3n+3})$

and

$G(t_{3n+3}, t_{3n+4}, t_{3n+5}) \leq M(x_{3n+2}, x_{3n+3}, x_{3n+4})$

$= G(t_{3n+2}, t_{3n+3}, z_{3n+4})$.

Hence, $\{G(t_k, t_{k+1}, t_{k+2})\}$ is a non-decreasing sequence of non-negative real numbers.

Thus, there is an $r \geq 0$ such that

$$\lim_{k\to\infty} G(t_k, t_{k+1}, t_{k+2}) = r. \tag{2.6}$$

Since

$G(t_{k+1}, t_{k+2}, t_{k+3}) \leq M(x_k, x_{k+1}, x_{k+2})$

$\leq G(t_k, t_{k+1}, t_{k+2})$,

as $k \to \infty$, we get

$$\lim_{k\to\infty} M(x_k, x_{k+1}, x_{k+2}) = r. \tag{2.7}$$

Letting $n \to \infty$ in (2.5), using (2.6), (2.7) and the continuity of ψ and ϕ, we get

$\psi(r) \leq \psi(r) - \phi(r) \leq \psi(r)$ and hence $\phi(r) = 0$. This gives us

$$\lim_{k\to\infty} G(x_k, x_{k+1}, x_{k+2}) = 0, \tag{2.8}$$

from our assumptions about ϕ. Also, from Definition 1.3 part (G-3), we have

$$\lim_{k\to\infty} G(x_k, x_{k+1}, x_{k+2}) = 0. \tag{2.9}$$

Step II. We will show that $\{t_n\}$ is a G-Cauchy sequence in X.

We will show that for every $\varepsilon > 0$, there exists a positive integer k such that for all $m, n \geq k$, $G(t_m, t_n, t_n) < \varepsilon$. Suppose the above statement is false. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{t_{m(k)}\}$ and $\{t_{n(k)}\}$ of $\{t_n\}$ such that $n(k) > m(k) \geq k$ and

(a) $m(k)=3t$ and $n(k) = 3t' + 1$, where t and t' are non-negative integers.

(b) $G(t_{m(k)}, t_{n(k)}, t_{n(k)}) \geq \varepsilon$. (2.10)

(c) $n(k)$ is the smallest number such that the condition (b) holds,

i.e. $G(t_{m(k)}, t_{n(k)-1}, t_{n(k)-1}) < \varepsilon$. (2.11)

From rectangle inequality and (2.11), we have

$G(t_{m(k)}, t_{n(k)}, t_{n(k)}) \leq G(t_{m(k)}, t_{n(k)-1}, t_{n(k)-1}) + G(t_{n(k)-1}, t_{n(k)}, t_{n(k)})$

$< \varepsilon + G(t_{n(k)-1}, t_{n(k)}, t_{n(k)+1})$.

As $k \to \infty$ and using (2.8) and (2.11), we have

$$\lim_{k\to\infty} G(t_{m(k)}, t_{n(k)}, t_{n(k)}) = \varepsilon. \tag{2.12}$$

Again from rectangle inequality,

$G(t_{m(k)}, t_{n(k)}, t_{n(k)+1}) \leq G(t_{m(k)}, t_{n(k)}, t_{n(k)}) + G(t_{n(k)}, t_{n(k)}, t_{n(k)+1})$

$\leq G(t_{m(k)}, t_{n(k)}, t_{n(k)}) + G(t_{n(k)}, t_{n(k)+1}, t_{n(k)+2})$

and

$G(t_{m(k)}, t_{n(k)}, t_{n(k)}) \leq G(t_{m(k)}, t_{n(k)}, t_{n(k)+1})$.

As $k \to \infty$, using (2.8), (2.10) and (2.12), we have

$$\lim_{k\to\infty} G\left(t_{m(k)}, t_{n(k)}, t_{n(k)+1}\right) = \varepsilon. \qquad (2.13)$$

On the other hand,

$G(t_{m(k)}, t_{n(k)+1}, t_{n(k)+1}) \leq G(t_{m(k)}, t_{n(k)}, t_{n(k)}) + G(t_{n(k)}, t_{n(k)+1}, t_{n(k)+1})$

and

$G(t_{n(k)}, t_{n(k)+1}, t_{m(k)}) \leq G(t_{n(k)}, t_{n(k)+1}, t_{n(k)+1}) + G(t_{n(k)+1}, t_{n(k)+1}, t_{m(k)})$.

As $k \to \infty$ and using (2.9), (2.12) and(2.13), we have

$$\lim_{k\to\infty} G\left(t_{m(k)}, t_{n(k)+1}, t_{n(k)+1}\right) = \varepsilon. \qquad (2.14)$$

In a similar way, we have

$G(t_{m(k)+1}, t_{n(k)}, t_{n(k)+1}) \leq G(t_{m(k)+1}, t_{m(k)}, t_{m(k)}) + G(t_{m(k)}, t_{n(k)}, t_{n(k)+1})$

$\leq 2G(t_{m(k)}, t_{m(k)+1}, t_{m(k)+1}) + G(t_{m(k)}, t_{n(k)}, t_{n(k)+1})$

and

$G(t_{m(k)}, t_{n(k)}, t_{n(k)+1}) \leq G(t_{m(k)}, t_{m(k)+1}, t_{m(k)+1}) + G(t_{m(k)+1}, t_{n(k)}, t_{n(k)+1})$ therefore, by taking limit $k\to\infty$ and using (2.9), (2.13), we get

$$\lim_{k\to\infty} G\left(t_{m(k)+1}, t_{n(k)}, t_{n(k)+1}\right) = \varepsilon. \qquad (2.15)$$

Also,

$G(t_{m(k)}, t_{n(k)+1}, t_{n(k)+1}) \leq G(t_{m(k)}, t_{m(k)+1}, t_{n(k)+1})$,

and

$G(t_{m(k)}, t_{m(k)+1}, t_{n(k)+1}) \leq G(t_{m(k)}, t_{m(k)+1}, t_{m(k)+1}) + G(t_{m(k)+1}, t_{m(k)+1}, t_{n(k)+1})$

$\leq G(t_{m(k)}, t_{m(k)+1}, t_{m(k)+1}) + G(t_{m(k)+1}, t_{n(k)}, t_{n(k)+1})$.

As $k\to\infty$ and using (2.9), (2.14), we have

$$\lim_{k\to\infty} G\left(t_{m(k)}, t_{m(k)+1}, t_{n(k)+1}\right) = \varepsilon. \qquad (2.16)$$

Also,

$G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+1}) \leq G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)})$ (2.17)

and

$G(t_{m(k)+1}, t_{n(k)}, t_{n(k)+1}) \leq G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+1}) + G(t_{n(k)+1}, t_{n(k)+1}, t_{n(k)})$.

(2.18)

So, from (2.9), (2.15), (2.16) and (2.17), we have

$$\lim_{k\to\infty} G\left(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+1}\right) = \varepsilon. \qquad (2.19)$$

Finally,

$G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+2}) \leq G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+1}) + G(t_{n(k)+1}, t_{n(k)+1}, t_{n(k)+2})$

$\leq G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+1}) + G(t_{n(k)}, t_{n(k)+1}, t_{n(k)+2})$

and

$$G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+1}) \le G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+2}).$$

As $k \to \infty$ and by using (2.8), (2.19), we have

$$\lim_{k\to\infty} G\left(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+2}\right) = \varepsilon. \tag{2.20}$$

Since $x_{m(k)} \preccurlyeq x_{n(k)} \preccurlyeq x_{n(k)+1}$, putting $x = x_{m(k)}$, $y = x_{n(k)}$, and $z = x_{n(k)+1}$ in (2.1) for all $k \ge 0$,we have

$$\psi(G(t_{m(k)+1}, t_{n(k)+1}, t_{n(k)+2})) = \psi(G(fx_{m(k)}, gx_{n(k)}, hx_{n(k)+1}))$$

$$\le \psi(M(x_{m(k)}, x_{n(k)}, x_{n(k)+1})) - \phi(M(x_{m(k)}, x_{n(k)}, x_{n(k)+1})),$$

where

$$M\left(x_{m(k)}, x_{n(k)}, x_{n(k)+1}\right) = \max\left\{\begin{array}{c} G\left(Tx_{m(k)}, Rx_{n(k)}, Sx_{n(k)+1}\right), G\left(Rx_{n(k)}, Sx_{n(k)+1}, hx_{n(k)+1}\right), \\ G\left(Tx_{m(k)}, gx_{n(k)}, gx_{n(k)}\right), \\ G\left(Rx_{n(k)}, hx_{n(k)+1}, hx_{n(k)+1}\right), G\left(Sx_{n(k)+1}, fx_{m(k)}, fx_{m(k)}\right), \\ G\left(Tx_{m(k)}, fx_{m(k)}, fx_{m(k)}\right), \\ G\left(Rx_{n(k)}, gx_{n(k)}, gx_{n(k)}\right), G\left(Sx_{n(k)+1}, Sx_{n(k)+1}, hx_{n(k)+1}\right) \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} G\left(t_{m(k)}, t_{n(k)}, t_{n(k)+1}\right), G\left(t_{n(k)}, t_{n(k)+1}, t_{n(k)+2}\right), \\ G\left(t_{m(k)}, t_{n(k)+1}, t_{n(k)+1}\right), \\ G\left(t_{n(k)}, t_{n(k)+2}, t_{n(k)+2}\right), G\left(t_{n(k)+1}, t_{m(k)+1}, t_{m(k)+1}\right), \\ G\left(t_{m(k)}, t_{m(k)+1}, t_{m(k)+1}\right), \\ G\left(t_{n(k)}, t_{n(k)+1}, t_{n(k)+1}\right), G\left(t_{n(k)+1}, t_{n(k)+1}, t_{n(k)+2}\right) \end{array}\right\}.$$

Taking $k \to \infty$ and using (2.9), (2.15), (2.20) we have

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon).$$

Hence, $\varepsilon = 0$, which is a contradiction. Consequently, $\{z_n\}$is a G-Cauchy sequence.

Step III. We will show that f, g, h, R, S and T have a common fixed point.

Since $\{t_n\}$is a G-Cauchy sequence in the complete symmetric G-metric spaceX, there exists $t \in X$ such that

$$\lim_{n\to\infty} G(t_{3n+1}, t_{3n+1}, t) = \lim_{n\to\infty} G(Rx_{3n+1}, Rx_{3n+1}, t) = \lim_{n\to\infty} G(fx_{3n}, fx_{3n}, t) = 0,$$

$$\lim_{n\to\infty} G(t_{3n+2}, t_{3n+2}, t) = \lim_{n\to\infty} G(Sx_{3n+2}, Sx_{3n+2}, t) = \lim_{n\to\infty} G(gx_{3n+1}, gx_{3n+1}, t) = 0$$

and

$$\lim_{n\to\infty} G(t_{3n+3}, t_{3n+3}, t) = \lim_{n\to\infty} G(Tx_{3n+3}, Tx_{3n+3}, t) = \lim_{n\to\infty} G(hx_{3n+2}, hx_{3n+2}, t) = 0$$

Suppose condition (i) of our theorem holds.

Assume that R and T are continuous and let the pairs (f, T) and (g, R) are compatible.

This implies that

$$\lim_{n\to\infty} G(Tfx_{3n}, fTx_{3n}, fTx_{3n}) = \lim_{n\to\infty} G(Tt, fTx_{3n}, fTx_{3n}) = 0,$$

and $\lim_{n\to\infty} G(Rgx_{3n+1}, gRx_{3n+1}, gRx_{3n+1}) = \lim_{n\to\infty} G(Rt, gRx_{3n+1}, gRx_{3n+1}) = 0.$

Since$Rx_{3n+1} \preccurlyeq fRx_{3n+1} \preccurlyeq x_{3n+1} \preccurlyeq gx_{3n+1}$

$= Sx_{3n+2} \preccurlyeq gSx_{3n+2} \preccurlyeq x_{3n+2} \preccurlyeq hx_{3n+2} = Tx_{3n+3}$,

by using (2.1) we obtain

$\psi(G(fTx_{3n+3}, gRx_{3n+1}, hx_{3n+2}))$

$$\leq \psi(M(Tx_{3n+3}, Rx_{3n+1}, x_{3n+2})) - \phi(M(Tx_{3n+3}, Rx_{3n+1}, x_{3n+2})), \quad (2.21)$$

where

$M(Tx_{3n+3}, Rx_{3n+1}, x_{3n+2})$

$$= \max \left\{ \begin{matrix} G(TTx_{3n+3}, RRx_{3n+1}, Sx_{3n+2}), G(RRx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\ G(TTx_{3n+3}, gRx_{3n+1}, gRx_{3n+1}), \\ G(RRx_{3n+1}, hx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, fTx_{3n+3}, fTx_{3n+3}), \\ G(TTx_{3n+3}, fTx_{3n+3}, fTx_{3n+3}), \\ G(RRx_{3n+1}, gRx_{3n+1}, gRx_{3n+1}), G(Sx_{3n+2}, Sx_{3n+2}, hx_{3n+2}) \end{matrix} \right\}.$$

Taking $n \to \infty$, in right hand side,

$M(Tx_{3n+3}, Rx_{3n+1}, x_{3n+2})$

$$= \max \left\{ \begin{matrix} G(Tt, Rt, t), G(Rt, t, t), G(Tt, Rt, Rt), G(Rt, t, tz), \\ G(t, Tt, Tt), G(Tt, Tt, Tt), G(Rt, Rt, Rt), G(t, t, t) \end{matrix} \right\}$$

$=G(Tt, R, z)$.

On taking the limit as $n \to \infty$in (2.21), we obtain that

$\psi(G(Tt, Rt, t)) \leq \psi(G(Tt, Rt, t)) - \phi(G(Tt, Rt, t))$,

and hence, $Tt = Rt = t$.

Since $x_{3n+1} \preccurlyeq x_{3n+2} \preccurlyeq hx_{3n+2}$and $hx_{3n+2} \to t$, as $n \to \infty$,

we have $x_{3n+1} \preccurlyeq x_{3n+2} \preccurlyeq t$. Therefore, from (2.1),

$\psi(G(ft, gx_{3n+1}, hx_{3n+2})) \leq \psi(M(t, x_{3n+1}, x_{3n+2})) - \phi(M(t, x_{3n+1}, x_{3n+2}))$, (2.22)

where $M(t, x_{3n+1}, x_{3n+2})$

$$= \max \left\{ \begin{matrix} G(Tt, Rx_{3n+1}, Sx_{3n+2}), G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\ G(Tt, gx_{3n+1}, gx_{3n+1}), \\ G(Rx_{3n+1}, hx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, ft, ft), \\ G(Tt, ft, ft), \\ G(Rx_{3n+1}, gx_{3n+1}, gx_{3n+1}), G(Sx_{3n+2}, Sx_{3n+2}, hx_{3n+2}) \end{matrix} \right\}.$$

Taking $n \to \infty$, in right hand side,

$$= \max \left\{ \begin{matrix} G(t, t, t), G(t, t, t), G(t, t, t), G(t, t, t), \\ G(t, ft, ft), G(t, ft, ft), G(t, t, t), G(t, t, t) \end{matrix} \right\}$$

$= G(ft, t, t)$.

Taking $n \to \infty$ in (2.22),we get

$$\psi(G(ft, t, t)) \leq \psi(G(ft, t, t)) - \phi(G(ft, t, t)),$$

hence $ft = t$.

Since $x_{3n+2} \preccurlyeq hx_{3n+2}$ and $hx_{3n+2} \rightarrow t$, as $n\rightarrow\infty$, we have $x_{3n+2} \preccurlyeq z$.

Hence from (2.1),

$$\psi(G(ft, gt, hx_{3n+2})) \leq \psi(M(t, t, x_{3n+2})) - \phi(M(t, t, x_{3n+2})). \qquad (2.23)$$

where

$$M(t, t, x_{3n+2}) = \max \begin{Bmatrix} G(Tt, Rt, Sx_{3n+2}), G(Rt, Sx_{3n+2}, hx_{3n+2}), G(Tt, gt, gt), \\ G(Rt, hx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, ft, ft), G(Tt, ft, ft), \\ G(Rt, gt, gt), G(Sx_{3n+2}, Sx_{3n+2}, hx_{3n+2}) \end{Bmatrix}$$

Taking $n\rightarrow \infty$, in right hand side,

$$M(t, t, x_{3n+2}) = \max \begin{Bmatrix} G(Tt, Rt, t), G(Rt, t, t), G(Tt, gt, gt), G(Rt, t, t), \\ G(t, ft, ft), G(Tt, ft, ft), G(Rt, gt, gt), G(t, t, t) \end{Bmatrix}$$

$= G(t, t, gt)$.

Making $n\rightarrow \infty$ in (2.23), we get

$\psi(G(t, gt, t)) \leq \psi(G(t, t, gt)) - \phi(G(t, t, gt))$,

which gives $gt = t$.

Since $g(X)$ is contained in $S(X)$, there exists a point $s \in X$ such that $t = gt = Sw$.

Suppose that $hw \neq Sw$.

Since $t \preccurlyeq gt = Sw \preccurlyeq gSw \preccurlyeq w$, we have $t \preccurlyeq w$. Hence, from (2.1),

$$\psi(G(ft, gt, hw)) \leq \psi(M(t, t, w)) - \phi(M(t, t, w)), \qquad (2.24)$$

where

$$M(t, t, w) = \max \begin{Bmatrix} G(Tt, Rt, Sw), G(Rt, Sw, hw), G(Tt, gt, gt), G(Rt, hw, hw), \\ G(Sw, ft, ft, G(Tt, ft, ft), G(Rt, gt, gt), G(Sw, Sw, hw)) \end{Bmatrix}$$

$= G(t, t, hw)$.

On taking the limit as $n\rightarrow\infty$ in (2.24), we obtain that

$\psi(G(t, t, hw)) \leq \psi(G(t, t, hw)) - \phi(G(t, t, hw))$,

which gives $hw = t$.

Since h and S are weakly compatible, we have $ht = hSw = Shw = St$.

Thus, t is a coincidence point of h and S. Now, we show that $ht = t$.

Since $x_{3n} \preccurlyeq fx_{3n}$ and $fx_{3n} \rightarrow t$, as $n\rightarrow\infty$, we have $x_{3n} \preccurlyeq t$. Hence, from (2.1),

$$\psi(G(fx_{3n}, gt, ht)) \leq \psi(M(x_{3n}, t, t)) - \phi(M(x_{3n}, t, t)), \qquad (2.25)$$

where

$$M(x_{3n}, t, t) = \max \begin{Bmatrix} G(Tx_{3n}, Rt, St), G(Rt, St, ht), G(Tx_{3n}, gt, gt), \\ G(Rt, ht, ht), G(St, fx_{3n}, fx_{3n}), G(Tx_{3n}, fx_{3n}, fx_{3n}), \\ G(Rt, gt, gt), G(St, St, ht) \end{Bmatrix}.$$

Taking $n\rightarrow \infty$, in right hand side,

$$M(x_{3n}, t, t) = \max \begin{Bmatrix} G(t, Rt, St), G(Rt, St, ht), G(t, gt, gt), G(Rt, ht, ht), \\ G(St, t, t), G(t, t, t), G(Rt, gt, gt), G(St, St, ht) \end{Bmatrix}$$

$= G(t, t, ht)$.

Letting $n\to\infty$ in (2.25), we obtain that

$\psi(G(t, t, ht)) \le \psi(G(t, t, ht)) - \phi(G(t, t, ht))$,

hence $ht = t$. Therefore, $ft = gt = ht = Rt = St = Tt = t$.

Following the same arguments, the result is true when (ii) or (iii) of our Theorem holds.

We claim that common fixed point off, g, h, R, S and T is unique. Assume on contrary that $fp = gp = hp = Rp = Sp = Tp = p$, $fq = gq = hq= Rq = Sq = Tq = q$ and $p \ne q$.

Using (2.1), we get

$\psi(G(fp, gq, hq)) \le \psi(M(p, q, q)) - \phi(M(p, q, q))$

where

$$M(p, q, q) = \max\begin{Bmatrix} G(Tp,Rq,Sq), G(Rq,Sq,hq), G(Tp,gq,gq), G(Rq,hq,hq), \\ G(Sq,fp,fp), G(Tp,fp,fp), G(Rq,gq,gq), G(Sq,Sq,hq) \end{Bmatrix}$$

$= G(p, q, q)$.

So,

$\psi(G(p, q, q)) \le \psi(G(p, q, q)) - \phi(G(p, q, q))$.

Therefore, $\phi(G(p, q, q)) = 0$ which implies that $p = q$.

References

[1]. M. Abbas , T. Nazir and S. Radenovic, Common fixed points of four maps in partially ordered metric spaces, *Appl. Math. Lett.,***24**(2011), p.1520–1526.

[2]. Ya. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, New results in operator theory, *Advances and Appl. 98 (ed. By I. Gohberg and Yu Lyubich), BirkhauserVerlag, Basel, 1997.*

[3]. B.C. Dhage, Generalized metric spaces and mappings with fixed point, *Bull. Cal.Math. Soc.,***84** (1992), p. 329-336.

[4]. G. Jungck, Compatible mappings and common fixed points, *Int. J. Math.Math. Sci. Vol.,***9** (1986), p. 771–779.

[5]. G. Jungck, Common fixed points for non-continuous non-self maps on nonmetric spaces, *Far East J. Math. Sci.,***4** (1996), p. 199–215.

[6]. M.S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.,***30**(1984), p. 1-9.

[7]. M . Kumar, Compatible Maps in G-Metric Spaces. *Int. Journal of Math. Anal.,* **6**(2012), p.1415– 1421.

[8]. Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *J.Nonlinear Convex Anal.,***7** (2006), p. 289-297.

[9]. B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Anal.,***47** (2001), p. 2683-2693.

[10]. Q. Zhang and Y. Song, Fixed point theory for generalized ϕ-weak contraction, *Appl. Math. Lett.,***22** (2009), p. 75-78.